

# Perfect Fluids: Field-theoretical Description and Gauge Symmetry Issue

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## Abstract

We show that combinations of (in general, non-linear) 2- and 3-form fields analogous to the Maxwell (1-form) field, completely describe perfect fluids, including the rotating ones. In the non-rotating case, the 2-form field is sufficient, and a free 3-form field proves to be equivalent to appearance of the cosmological term in Einstein's equations (the square-root non-linearity corresponding to  $\Lambda = 0$ ). The gauge degrees of freedom break down when a rotation is included, but even when they exist, there obviously fails to be realized an equivalence of the 2-form field and the massless scalar one recently claimed by Weinberg.

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We consider here  $r$ -form fields ( $r = 0, 1, 2$  and  $3$ , the corresponding  $r$ -forms for potentials being  $\varphi$ ,  $A$ ,  $B$ , and  $C$ ). Let the Lagrangian density  $\mathfrak{L} = \sqrt{-g}L$  be a function of Maxwell-type invariants ( $H = *(\Phi \wedge *\Phi)$ , *etc.* for  $I$ ,  $J$  and  $H$ ) of the corresponding field tensors, the  $(r+1)$ -forms  $\Phi = d\varphi$ ,  $F = dA$ ,  $G = dB$ , and  $H = dC$ . Thus the Lagrangian depends on the metric coefficients only algebraically. We use the spacetime signature  $+, -, -, -$ , and 4-dimensional Greek indices,  $*$  being the Hodge star. Below the dependence on Maxwell's field  $F$  will be omitted.

The 2nd Noether theorem (see [1]) gives in this case the standard definition of stress-energy tensor

$$\mathfrak{T}^{\mu\nu} = \sqrt{-g}T^{\mu\nu} = -2\frac{\partial\mathfrak{L}}{\partial g_{\mu\nu}} \quad (1)$$

(only the variational derivative with respect to  $g_{\mu\nu}$  is reduced to the partial one), so that

$$T = -Lg + 2H\frac{\partial L}{\partial H} \stackrel{(0)}{u} \times \stackrel{(0)}{u} + 2J\frac{\partial L}{\partial J} \left( g - \stackrel{(2)}{u} \times \stackrel{(2)}{u} \right) + 2K\frac{\partial L}{\partial K} g \quad (2)$$

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where  ${}^{(0)}u = \Phi/\sqrt{-H}$  and  ${}^{(2)}u = \tilde{G}/\sqrt{J}$ , while  $\tilde{G}_\alpha = (1/3!)G^{\lambda\mu\nu}E_{\lambda\mu\nu\alpha}$ ;  $E_{\kappa\lambda\mu\nu} := \sqrt{-g}\epsilon_{\kappa\lambda\mu\nu}$ ,  $\epsilon_{\kappa\lambda\mu\nu}$  being the Levi-Civita symbol.

The phenomenological stress-energy tensor of a perfect fluid has in our notations the form  $T^{\text{Pf}} = (\mu + p)u \otimes u - pg$ . Here  $p$  is invariant pressure of the fluid,  $\mu$  its invariant mass (energy) density, and  $u$  its local four-velocity. Thus a stress-energy tensor acceptable for description of perfect fluids, should have one distinct eigenvalue  $\mu$  corresponding to the eigenvector  $u$  and another (now, triple) eigenvalue  $(-p)$  corresponding to any vector of the whole local subspace orthogonal to  $u$ . One may find information about the main stages of development of the theory of perfect fluids in [2], [3], [4], [5]; this paper was first published in [8] in a somewhat shorter form; also see a much more detailed paper [9] (though without gauge-related considerations).

Let us first consider the pure free field case: only one of the  $r$ -form fields should be then present at once, or  $L = {}^{(0)}L + {}^{(2)}L + {}^{(3)}L$ , the consecutive terms depending on the corresponding  $r$ -form field variables. The tensor (2) is compatible with the above conditions for the cases  $r = 0, 2$  or  $3$  only (Maxwell's field and its non-linear analogues do not meet the requirements, hence they were already omitted). A comparison with (2) yields  $\mu = 2H \partial {}^{(0)}L / \partial H - {}^{(0)}L$ ,  $p = L$ ;  $\mu = - {}^{(2)}L$ ,  $p = L - 2J \partial {}^{(2)}L / \partial J$ ;  $\mu = - {}^{(3)}L$ ,  $p = - {}^{(3)}L + 2K \partial {}^{(3)}L / \partial K$ . It is also clear that the vector  ${}^{(0)}u$  is timelike (thus suitable for description of four-velocity), only if the scalar field  $\varphi$  is essentially non-stationary, but for  ${}^{(2)}u$  there is the exactly opposite situation: in the 2-form field potential the  $t$ -dependence has not to dominate, or it even may be absent (for a static or stationary 2-form field). Moreover, the  $p = 0$  case (incoherent dust) cannot be described at all via the scalar field, in contrast to the 2-form field. Thus one has to exclude the 0-form (massless scalar) field if the problem under consideration is to describe a perfect fluid which can be brought to its rest frame (at least locally). Therefore the superscript (2) in  ${}^{(2)}u$  will henceforth be omitted.

In the case of a pure 3-form field, the stress-energy tensor (2) is explicitly proportional to  $\delta_\alpha^\beta$ : it is equivalent to addition of a cosmological term to Einstein's equations. From (2) it is also obvious that  $T_\alpha^\beta$  identically vanishes when  $L \sim \sqrt{K}$ . This latter case can be called that of a phantom 3-form field which may be described by an *arbitrary* function of coordinates. In the former case, the 3-form field is simply constant (we speak on only one function since everything is determined here by a pseudoscalar, the dual conjugate to the 4-form  $H$ ). The both cases follow also from the field equations being a result of variational principle applied to  $L$ ; though the 3-form field is non-dynamical in this sense, it affects the global geometry of the universe via determination of the cosmological term, and it may provide virtual particles in quantum theoretical Feynman-type graphs, when coupled to other fields. Thus one could relate this field to the fundamental cosmological field proposed by Sakurai [6] (another reason is its decisive role in description of rotating fluids, especially when one

thinks about the global aspects of rotation and the Mach principle).

Weinberg [7] has given a generalization of the gauge field theory (essentially of the electromagnetic field) to the case of  $p$ -form fields (Section 8.8 of his very instructive and well written book; we have to speak here about the  $r$ -form fields simply because  $p$  means pressure of the fluid in our context). Weinberg's main conclusion in this respect was that 'in four spacetime dimensions,  $p$ -forms offer no new possibilities':  $p = 3$  is simply an empty case, and  $p = 2$  'is equivalent to a zero-form gauge field, which as we have seen is equivalent to a massless derivatively coupled scalar field'. Our communication however represents a counterexample to these conjectures, as it is seen from the last two paragraphs above. The error committed by Weinberg was that the physical interpretation of  $p$ -form fields (as well as all other geometric objects playing rôles of physical fields) does not reduce to their geometric properties, but it crucially depends on their dynamical equations, *i.e.* the structure of corresponding Lagrangians.

In the pure 2-form field case, it is easy to translate into the field theoretical language all general relativistic solutions for non-rotating fluid (for all cases of linear dependence of  $p$  on  $\mu$ , as well as for polytrope equations of state; the only known exception is the interior Schwarzschild solution which can be translated in the context of interacting 2- and 3-form fields).

In the case of a pure 2-form field, the field equations reduce to

$$d\left(J^{1/2}\frac{dL}{dJ}u\right) = 0, \quad (3)$$

$u$  being the normalized 1-form of  $\tilde{G}$  (four-velocity of the fluid). One finds immediately that the fluid does not rotate. The only remedy is in this case an introduction of a source term in (3).

The simplest way to do this is to introduce in the Lagrangian density dependence on a new invariant  $J_1 = -B_{[\kappa\lambda}B_{\mu\nu]}B^{[\kappa\lambda}B^{\mu\nu]}$  which does not spoil the structure of stress-energy tensor (alongside with  $J_1$ , we shall use the old invariants  $J$  and  $K$ ). Since

$$B_{[\kappa\lambda}B_{\mu\nu]} = -\frac{2}{4!}B_{\alpha\beta}B^{\alpha\beta}{}^*E_{\kappa\lambda\mu\nu} \quad (4)$$

where  $B^{\alpha\beta}{}^* := \frac{1}{2}B_{\mu\nu}E^{\alpha\beta\mu\nu}$  (dual conjugation),  $J_1^{1/2} = 6^{-1/2}B_{\alpha\beta}B^{\alpha\beta}{}^*$ . In fact,  $J_1 = 0$ , if  $B$  is a simple bivector (this corresponds to all types of rotating fluids discussed in existing literature). This does not however annul the expression which this invariant contributes to the 2-form field equations: it is proportional to  $\partial J_1^{1/2}/\partial B_{\mu\nu} \neq 0$ . Thus let the Lagrangian density be

$$\mathfrak{L} = \sqrt{-g}\left(L(J) + M(K)J_1^{1/2}\right). \quad (5)$$

The 2-form field equations now take the form

$$d\left(\frac{dL}{dJ}\tilde{G}\right) = \sqrt{2/3}M(K)B. \quad (6)$$

In their turn, the 3-form field equations yield the first integral

$$J_1^{1/2} K^{1/2} \frac{dM}{dK} = \text{const} \equiv 0 \quad (7)$$

(since  $J_1 = 0$ ) in agreement with the fact that  $K$  (hence,  $M$ ) is an *arbitrary* function, if only the 3-form field equations are taken into account. Though the equations (6) apparently show that the  $\tilde{G}$  congruence should in general be rotating, the 2-form field  $B$  is an exact form for solutions with constant  $M(K)$ , thus its substitution into the left-hand side of (6) via  $\tilde{G}$ , leads trivially to vanishing of  $G$  (and hence  $B$ ). We see that in a non-trivial situation the cosmological field  $K$  has to be essentially non-constant.

But the complete set of equations contains Einstein's equations as well. One has to take into account their sources and the structure of their solutions in order to better understand this remarkable situation probably never encountered in theoretical physics before.

The gauge freedom suggested by  $G = dB$ , is obviously destroyed by the field equations (6) when a rotation is switched on. Since the rotation is so widespread in nature, existence of the gauge freedom in  $B$  should merely be an exclusion than a rule.

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